MATH 312 Lecture 10 | UNDERSTANDING THE FOUR FUNDAMENTAL SPACES

Last time: Given $m \times n$ matrix $A: \mathbb{R}^n \to \mathbb{R}^m$, we looked at 4 subspaces:

- $N(A)$ and $C(A^T)$ are subspaces of $\mathbb{R}^n$,
- $C(A)$ and $N(A^T)$ are subspaces of $\mathbb{R}^m$

Today, we see how to recover these spaces (or, "bases of these spaces") from the reduced row echelon form, in a simple example:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \quad A: \mathbb{R}^3 \to \mathbb{R}^2.$$

So: $n = 3$, $m = 2$ and $A: \mathbb{R}^3 \to \mathbb{R}^2$.

- To put A in RREF, it takes only one operation: $R_2' = R_2 - 2R_1$ which has matrix $M = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. So,

$$MA = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} = \begin{bmatrix} \overset{\text{(pivot)}}{1} & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R, \text{ red row echelon}$$

- Now, M is invertible: $M^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, so

$$A = M^{-1}R = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} R$$

So, $\boxed{\text{Rank}(A) = 1.}$

We can easily "see" that:

$$C(A) = \text{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \;\&\; C(A^T) = \text{span}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

even without any RREF! But for more complicated matrices, RREF is the best strategy.

Anyway here's what we have:

$$A = M^{-1}R$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

I — C(A)

Now, $C(A)$ is clearly **NOT** the same as $C(R)$! $C(R)$ always has second component $= 0$. But on the other hand, the matrix M^{-1} sends $C(R)$ to $C(A)$!!

Now, $C(R) =$ span of columns of R others depend linearly on these

$=$ span of pivot columns of R

$=$ span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This line!

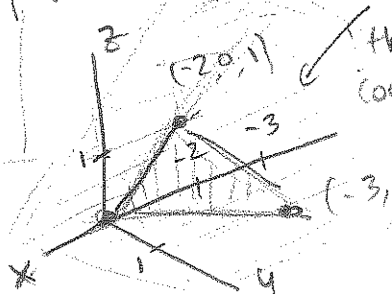
So, $C(R)$ has basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Now, $C(A)$ has basis $= M^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$\dim C(A) = 1$.

II — N(A)

$A = M^{-1}R$, and since M^{-1} is invertible, it only sends zero to zero. So, $N(A) = N(R)$ and of course $N(R)$ is easily computed! The free variables are "y" and "z", and $x + 3y + 2z = 0$, so: $N(A)$ is of the form $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z$, and hence has basis $= \boxed{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$



Plane across the triangle (0,0,0), (-2,0,1), (-3,1,0)

$\dim N(A) = 2$.

(III) $C(A^T)$

This is the SPAN of the ROWS of A. Since M^{-1} only produces linear combinations of these rows, (In this case, $R_2' = R_2 + 2R_1$) in an INVERTIBLE way, the row space of R is the SAME as that of A,

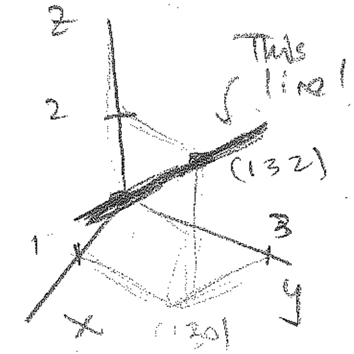
$$C(A^T) = C(R^T)$$

And of course, the zeroed-out rows in R don't contribute ANYTHING to the span! So,

$$C(A^T) = \text{span}\begin{pmatrix}1\\3\\2\end{pmatrix}$$ → pivot row.

So a basis for $C(A^T)$ is $\boxed{\begin{pmatrix}1\\3\\2\end{pmatrix}}$

$$\dim(C(A^T)) = 1.$$



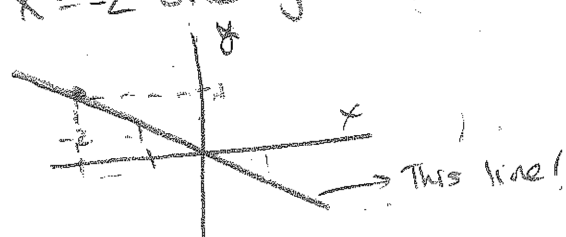
(IV) $N(A^T)$

This is the Null space of $A^T = \begin{bmatrix}1 & 2\\3 & 6\\2 & 4\end{bmatrix}$, a subspace of $\mathbb{R}^2$, i.e. all $\begin{pmatrix}x\\y\end{pmatrix}$ so that

$$A^T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+2y\\3x+6y\\2x+4y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

But the left side is $\begin{pmatrix}1\\3\\2\end{pmatrix}x + \begin{pmatrix}2\\6\\4\end{pmatrix}y$, so we want ALL LINEAR COMBS of the rows of A that give zero! We already see: IN $R = \begin{bmatrix}1 & 3 & 2\\0 & 0 & 0\end{bmatrix}$ and $M = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}$! So, to get $(0\ 0\ 0)$, set $x=-2$ and $y=1$.

$$N(A^T) = \text{span}\begin{pmatrix}1\\-2\end{pmatrix},$$

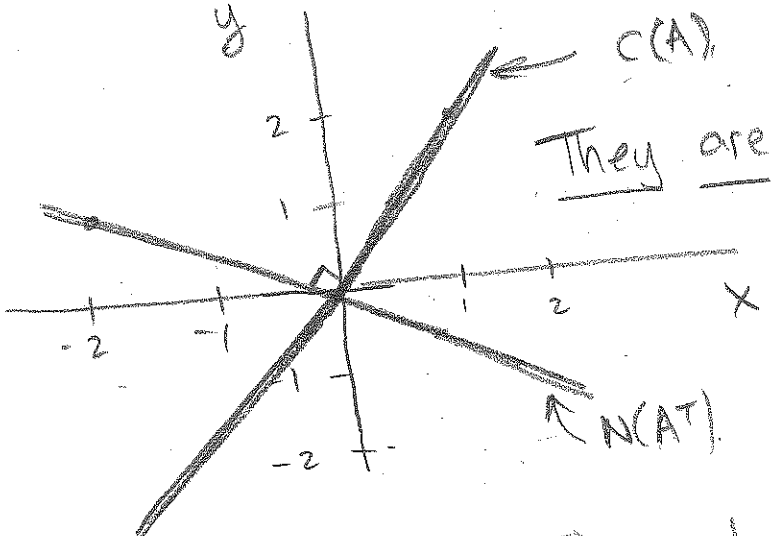


dim = 1, basis = $\begin{bmatrix}-2\\1\end{bmatrix}$

SOME GEOMETRY

Here are two pictures:

- In $\mathbb{R}^2$, we have $C(A)$ and $N(A^T)$, each of dimension 1:

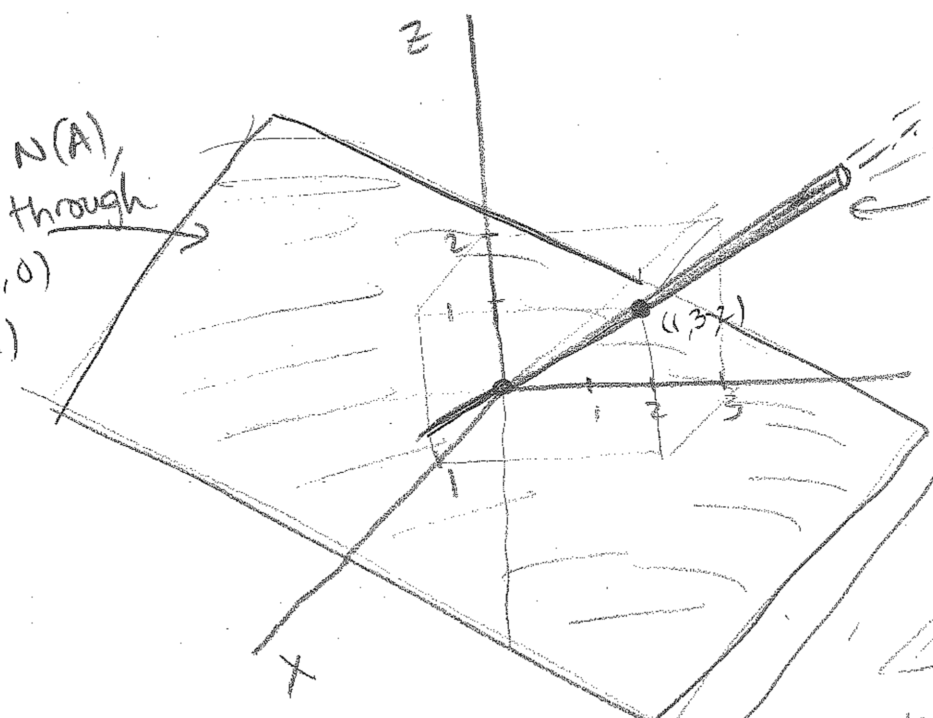


They are perpendicular!

- In $\mathbb{R}^3$, we have $C(A^T)$ and $N(A)$, of dimensions 1 and 2 respectively:

This is $N(A)$, the plane through $(0,0,0)$, $(-3,1,0)$ and $(-2,0,1)$

(Hard to draw).



These are ALSO perpendicular